Sheaves on Stratified Spaces

joint with Jens Eberhardt, Catharina Stroppel } Uni Bonn

Setup

$$\begin{array}{c} \{\lambda \neq 0\} \\ \shortparallel \\ \mathbb{G}_m = \mathbb{C}^\times \end{array} \xhookrightarrow{\text{acts on}} Z$$

Z: smooth projective complex variety

Fixed points / Attractor / Repeller

$$\underset{X}{\overset{\cup}{Z^0}} = Z^{\mathbb{G}_m} = \{ z \in Z \mid \lambda z = z \}$$

$\leftarrow$ $\boxed{\text{We assume } Z^0 \text{ discrete}}$

$$T_x Z = \bigoplus_{n \in \mathbb{Z}} \underbrace{(T_x Z)_n}_{\{v \in T_x Z \mid \lambda \cdot v = \lambda^n v\}} = (T_x Z)_{>0} \oplus (T_x Z)_{<0}$$

$$\left.\begin{array}{ll} Z_x^+ = \{ z \in Z \mid \lim_{\lambda \to 0} \lambda z = x \} \cong (T_x Z)_{>0} & \text{attractor} \\ Z_x^- = \{ z \in Z \mid \lim_{\lambda \to \infty} \lambda z = x \} \cong (T_x Z)_{<0} & \text{repeller} \end{array}\right\} \text{affine spaces}$$

$$Z = \bigcup_{x \in Z^0} Z_x^+ = \bigcup_{x \in Z^0} Z_x^- \qquad \text{Białynicki-Birula stratification}$$

Sheaves:

derived category of sheaves on $Z^{an}(\mathbb{C})$

$$\mathbb{D}_{Z^+}(Z) = \{ \mathcal{F} \in \mathbb{D}^b(Z) \mid \mathcal{F}|_{Z^+_\alpha} \text{ constant } \forall x \in Z^\circ \}$$

stratified sheaves

$$= \langle \Delta_x = \underline{\mathbb{Z}}_{Z^+_x} \mid x \in Z^\circ \rangle$$

standard object

extension by 0

"standard extension algebra"

$$E = \bigoplus_{x,y \in Z^\circ} \operatorname{Ext}^\bullet(\Delta_x, \Delta_y)$$

? ? ? ? ?

Why bother?

(1) Standard extension algebra (+ A_∞/dg-structure) encodes

how $D_{Z^+}(Z)$ is glued from $\{D_{const}(Z_x^+) = D^b(\mathbb{Z}\text{-mod})\}_{x \in Z^0}$:

easy

$$D_{Z^+}(Z) = \text{"} D_{perf}(\tilde{E}) \text{"}$$

E + higher structure

intersection cohomology complex

(2) Usually, $D_{Z^+}(Z)$ is studied via $IC(\overline{Z}_x^+) \in Perv(Z)$.

Many open questions on standard extension algebra remain!

Why bother?

(3) Main example: $Z = G/P$ for $P \subset G$

partial flag variety

parabolic

reductive/Kac-Moody group

Lie algebras of $P \subset G$

$$D_{Z^+}(Z) \cong D^b(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}))$$

parabolic category $\mathcal{O}$

$\Delta_X \longleftrightarrow$ Verma module

$E \longleftrightarrow$ Extensions of Verma modules

notoriously difficult

$\dim E \longleftrightarrow R'$-polynomial (related to R-polynomial)

Why bother?

(4) Categories of the form $D_{Zf}(Z)$ appear in many contexts in geometric representation theory, geometric Langlands ...

(5) Connection to Fukaya categories + Categorification of tensor products (Ask Catharina!)

Theorems

composition

Thm A (Kazhdan-Lusztig, Riche-Soergel-Williamson, E.-Stroppel):

There is a geometric model for the standard extension algebra

$$(E, \circ) \cong \Big(\bigoplus_{x,y\in Z^0} H_c^\bullet(Z_x^- \cap Z_y^+), *\Big)$$

composition of extensions cohomology with compact support convolution product $\square$

Theorems

Let $Z = G/P$

$\rightsquigarrow$ $Z_x^- \cap Z_y^+ =$ open Richardson variety (Kazhdan-Lusztig)

Thm.B (E.-Stroppel) There is a multiplication

$$\psi : Z_x^- \cap Z_y^+ \times Z_y^- \cap Z_z^+ \longrightarrow Z_x^- \cap Z_z^+ ,$$ such that

$$(E, \circ) \cong \Big(\bigoplus_{x,y \in Z^0} H_c^\bullet(Z_x^- \cap Z_y^+), *_\psi \Big)$$

product using ψ

$\square$

Theorems

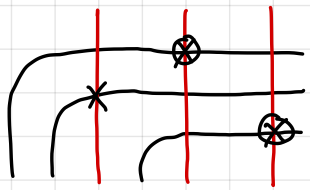
Let $Z = Gr(n,k) =$ k-dim. subspaces of $\mathbb{C}^n$

Thm. C (E.-Stroppel) There is a Deodhar-type decomposition on $Z_x^- \cap Z_y^+$ which is a stratification with strata $\cong \mathbb{G}_m^a \times \mathbb{A}^b$ $\square$

Remarkable since usually Deodhar decomposition is not a stratification.

Thm. D (E.-Stroppel) There is a diagrammatic calculus encoding

- Deodhar-type decomposition
- model for E



⋮

Part I

Examples!

Example 1 : $\mathbb{P}^1$

$$\lambda \cdot (z_0 : z_1) = (z_0 : \lambda^{-1} z_1) = (\lambda z_0 : z_1)$$

$$Z^0 = \{ e_0 = (1:0) , e_1 = (0,1) \}$$

$Z_0^- = \{(1:*)\} \cong \mathbb{A}^1$ $\quad$ $Z_1^- = \{e_1\}$

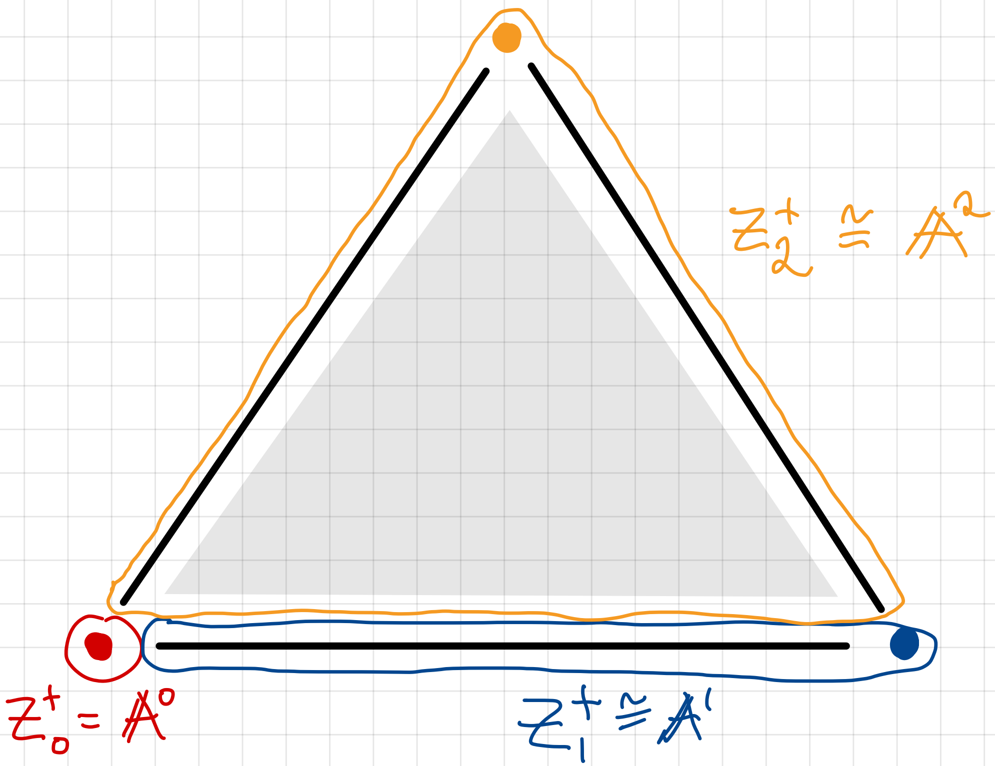
$\mathbb{P}^1$:

$Z_0^+ = \{e_0\}$ $\quad$ $Z_1^+ = \{(*:1)\} \cong \mathbb{A}^1$

Example 2 : $\mathbb{P}^2$

$$\lambda \cdot (z_0 : z_1 : z_2) = (z_0 : \lambda^{-1} z_1 : \lambda^{-2} z_2) = (\lambda^2 z_0 : \lambda z_1 : z_2)$$

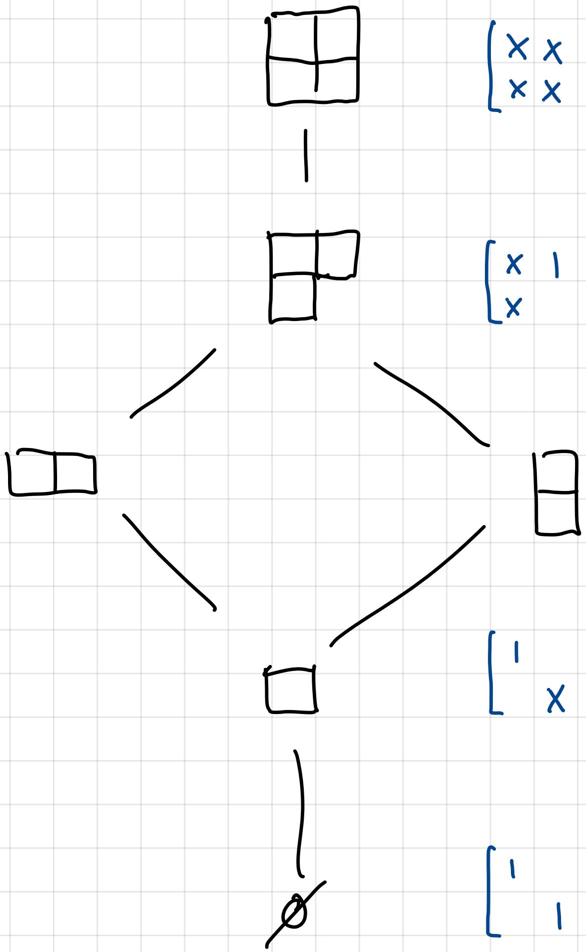
$$Z^0 = \{ e_0 = (1:0:0),\ e_1 = (0:1:0),\ e_2 = (0:0:1) \}$$



Example 3: $Gr(2,4) = \{ V \subset \mathbb{C}^4 \mid \dim V = 2 \}$

$$\lambda\,(x_1, x_2, x_3, x_4) = (x_1, \lambda^{-1}x_1, \lambda^{-2}x_2, \lambda^{-3}x_3)$$

Attractors:



$\begin{bmatrix} x & x & 1 & \\ x & x & & 1 \end{bmatrix} \cong \mathbb{A}^4$

$\begin{bmatrix} x & 1 & & \\ x & & x & 1 \end{bmatrix} \cong \mathbb{A}^3$

$\mathbb{A}^2 = \begin{bmatrix} 1 & & & \\ & x & x & 1 \end{bmatrix}$

$\begin{bmatrix} x & 1 & & \\ x & & 1 & \end{bmatrix} \cong \mathbb{A}^2$

$\begin{bmatrix} 1 & & & \\ & x & 1 & \end{bmatrix} \cong \mathbb{A}^1$

$\begin{bmatrix} 1 & & & \\ & 1 & & \end{bmatrix} \cong \mathbb{A}^0$

Extensions via cohomology:

$\boxed{\text{Thm A:}}$

proof: later

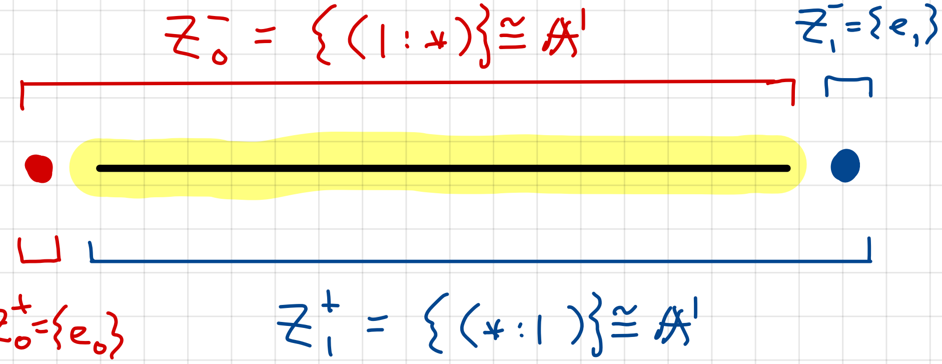
$$\mathrm{Ext}^\bullet(\Delta_x, \Delta_y) \cong H_c^\bullet(Z_x^- \cap Z_y^+)$$

Example 1 : $\mathbb{P}^1$ (cont.)

$$\lambda \cdot (z_0 : z_1) = (z_0 : \lambda^{-1} z_1)$$

$$Z^0 = \{ e_0 = (1:0) ,\ e_1 = (0,1) \}$$

$\mathbb{P}^1$:



$$\mathrm{Ext}^i(\Delta_0, \Delta_1) = H_c^i(Z_0^- \cap Z_1^+) = \begin{cases} \mathbb{Z} & i = 1, 2 \\ 0 & \text{else} \end{cases}$$

$$Z_0^- \cap Z_1^+ = \{(x:y) \mid x, y \neq 0\} = \mathbb{G}_m$$

Example 3 $\quad Gr(2,4) = \{ V \subset \mathbb{C}^4 \mid \dim V = 2 \}$ (cont.)

$$\lambda\,(x_1, x_2, x_3, x_4) = (x_1, \lambda^{-1} x_1, \lambda^{-2} x_2, \lambda^{-3} x_3)$$

$\boxplus \quad Z^-_{\boxplus} = \begin{bmatrix} & & 1 & \\ & & & 1 \end{bmatrix} \quad Z^+_{\boxplus} = \begin{bmatrix} x & x & 1 & \\ x & x & & 1 \end{bmatrix}$

$\varnothing \quad Z^-_{\varnothing} = \begin{bmatrix} 1 & & x & x \\ & 1 & x & x \end{bmatrix} \quad Z^+_{\varnothing} = \begin{bmatrix} 1 & & & \\ & 1 & & \end{bmatrix}$

$$Z^-_{\varnothing} \cap Z^+_{\boxplus} = \begin{bmatrix} 1 & & x & x \\ & 1 & x & x \end{bmatrix} \cap \begin{bmatrix} x & x & 1 & \\ x & x & & 1 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} 1 & & \\ & 1 & A \end{bmatrix} \;\middle|\; \exists B \in GL_2(\mathbb{C}) : B \begin{bmatrix} 1 & & \\ & 1 & A \end{bmatrix} = \begin{bmatrix} & & 1 & \\ C & & & 1 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & & \\ & 1 & A \end{bmatrix} \;\middle|\; A \in GL_2(\mathbb{C}) \right\} \cong GL_2(\mathbb{C})$$

$$\mathrm{Ext}^i(\Delta_{\varnothing}, \Delta_{\boxplus}) = H^i_c(Z^-_{\varnothing} \cap Z^+_{\boxplus}) = H^i_c(GL_2(\mathbb{C}))$$

$$= \begin{cases} \mathbb{Z} & 5 \leq i \leq 8 \\ 0 & \text{else} \end{cases}$$

Towards Thm A

$$Z^+ = \biguplus Z_x^+ \quad , \quad Z^- = \biguplus Z_x^-$$

$$\begin{tikzcd}
Z^0 \arrow[r, shift right, "c^+"'] \arrow[d, shift left, "i^-"] & Z^+ \arrow[l, shift right, "q^+"'] \arrow[d, "p^+"] \\
Z^- \arrow[u, shift left, "q^-"] \arrow[r, "p^-"'] & Z
\end{tikzcd}$$

$$\Delta = (p^+)_! (q^+)^* \underline{\mathbb{Z}}_{Z^0} = \bigoplus_{x \in Z^0} \Delta_x$$

$$E = \mathrm{Ext}^\bullet(\Delta, \Delta)$$

Braden's hyperbolic localization

$$\begin{tikzcd} Z^0 \arrow[r, shift left, "q^+"] \arrow[d, shift left, "i^-"] & Z^+ \arrow[l, shift left, "i^-"] \arrow[d, "p^+"] \\ Z^- \arrow[u, shift left, "q^-"] \arrow[r, "p^-"'] & Z \end{tikzcd}$$

$$(q^-)_* (p^-)^! \simeq (q^+)_! (p^+)^* : D_{mon}(Z) \to D(Z^0)$$

$$D_{mon}(Z) = \{ \mathcal{F} \in D(Z) \mid act^* \mathcal{F} \cong p^* \mathcal{F} \}$$

$$\mathbb{G}_m \times Z \overset{act^*}{\underset{p^*}{\rightrightarrows}} Z$$

Proof:

$$
\begin{aligned}
\operatorname{Ext}^\bullet(\Delta, \Delta) &= \operatorname{Ext}^\bullet\big((p^+)_!(q^+)^* Z, (p^+)_!(q^+)^* Z\big) \\
&= \operatorname{Ext}^\bullet\big(Z, (q^+)_*(p^+)^! (p^+)_!(q^+)^* Z\big) \\
&= \operatorname{Ext}^\bullet\big(Z, (q^-)_!(p^-)^* (p^+)_!(q^+)^* Z\big) \\
&= \operatorname{Ext}^\bullet\big(Z, (q^-)_!(k^+)_!(k^-)^*(q^+)^* Z\big) \\
&= \operatorname{Ext}^\bullet\big(Z, \pi_! \pi^* Z\big) \\
&= H_c^\bullet\big(Z^- \times_Z Z^+\big)
\end{aligned}
$$

$$
\begin{tikzcd}
Z^0 & & \\
& Z^- \times_Z Z^+ \arrow[ul, "\pi"'] \arrow[r, "k^-"] \arrow[d, "k^+"] & Z^+ \arrow[ull, bend right, "q^+"'] \arrow[d, "p^+"] \\
& Z^- \arrow[uul, bend left, "q^-"] \arrow[r, "p^-"'] & Z
\end{tikzcd}
$$

Composition?

$$
\begin{tikzcd}
\mathrm{Ext}^\bullet(\Delta,\Delta) \arrow[d, "\wr"] & \otimes & \mathrm{Ext}^\bullet(\Delta,\Delta) \arrow[d, "\wr"] \arrow[r] & \mathrm{Ext}^\bullet(\Delta,\Delta) \arrow[d, "\wr"] \\
H_c^\bullet(Z^- \times_Z Z^+) & \otimes & H_c^\bullet(Z^- \times_Z Z^+) \arrow[r, "???", "???"'] & H_c^\bullet(Z^- \times_Z Z^+)
\end{tikzcd}
$$

Drinfeld-Gaitsgory's Interpolation Space

$$\begin{tikzcd} \tilde{Z} \arrow[d] & \subset & \mathbb{A}^1 \times Z \times Z \arrow[dll] \\ \mathbb{A}^1 & & \end{tikzcd}$$

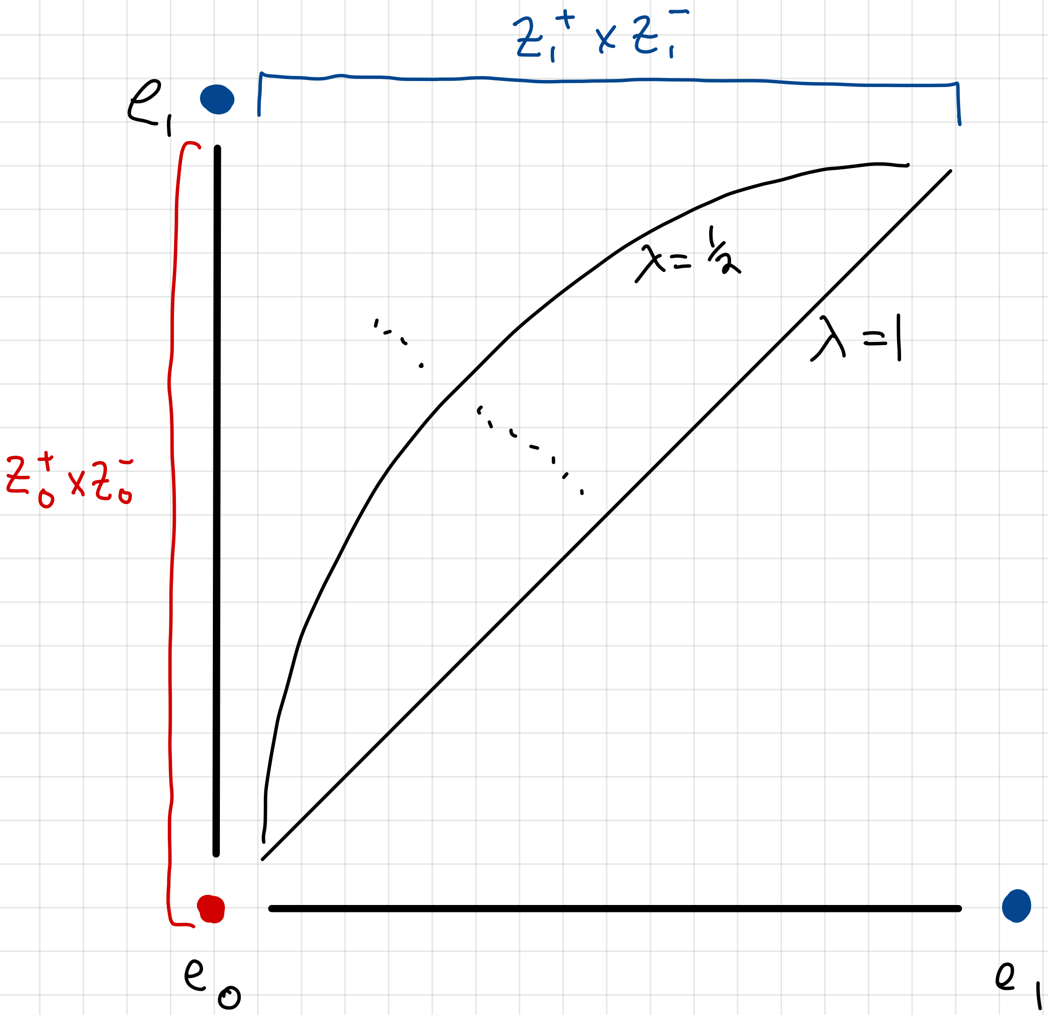
$$\tilde{Z}|_{\mathbb{G}_m} = \{(\lambda, z, \lambda z)\} = \text{graph of } \mathbb{G}_m\text{-action}$$

$$\tilde{Z}|_{\{1\}} = \{1\} \times \Delta Z \cong Z$$

can degenerate Z to $Z^+ \times_{Z^0} Z^-$

$$\tilde{Z}|_{\{0\}} = \{0\} \times Z^+ \times_{Z^0} Z^- = \{0\} \times \biguplus_{x \in Z^0} Z^+_x \times \underset{\substack{\cong \\ T_x Z}}{Z^-_x}$$

Example $\mathbb{P}^1$



Composition!

$$\begin{tikzcd}
Z^- \times_Z Z^+ \times_{Z^0} Z^- \times_Z Z^+ \arrow[r] \arrow[d] & Z^- \times_Z \tilde{Z} \times_Z Z^+ \arrow[d] & Z^- \times_Z Z \times_Z Z^+ = Z^- \times_Z Z^+ \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{A}^1 & \{1\} \arrow[l]
\end{tikzcd}$$

$$H_c^\bullet(Z^- \times_Z Z^+) \otimes H_c^\bullet(Z^- \times_Z Z^+) \xrightarrow{\boxtimes} H_c^\bullet(Z^- \times_Z Z^+ \times_{Z^0} Z^- \times_Z Z^+)$$

$$\xrightarrow{\mathrm{cosp}} H_c^\bullet(Z^- \times_Z Z^+) \quad \rightsquigarrow \boxed{\text{Thm. A}}$$

Towards Thm B

$Z = G/P \rightsquigarrow Z_x^- \cap Z_y^+ =$ open Richardson variety (Kazhdan-Lusztig)

Can get rid of $\tilde{Z}$ via explicit multiplication

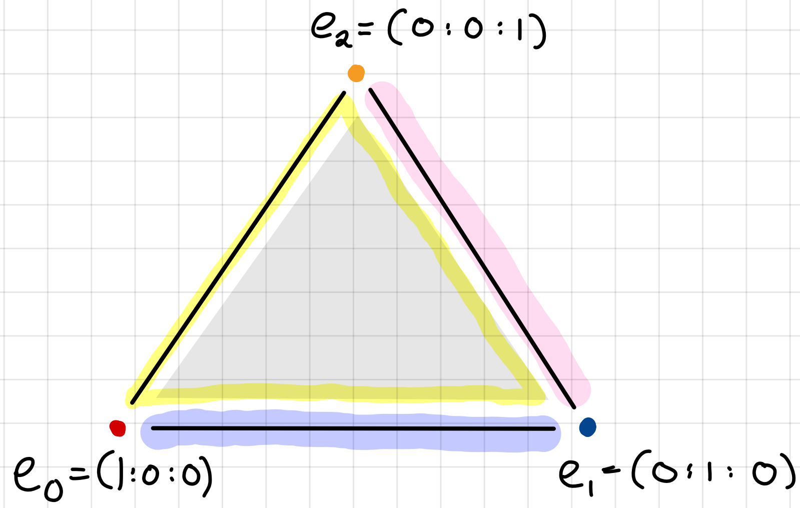
$$\psi : Z_x^- \cap Z_y^+ \times Z_y^- \cap Z_z^+ \hookrightarrow Z_x^- \cap Z_z^+$$

open embedding

$$H_c^\bullet(Z^- \times_Z Z^+) \otimes H_c^\bullet(Z^- \times_Z Z^+) \xrightarrow{\boxtimes} H_c^\bullet(Z^- \times_Z Z^+ \times_{Z^\circ} Z^- \times_Z Z^+)$$

$$\xrightarrow{\psi_!} H_c^\bullet(Z^- \times_Z Z^+) \rightsquigarrow \boxed{\text{Thm. B}}$$

Example $\mathbb{P}^2$ (cont.)



$$
\begin{tikzcd}
(Z_0^- \cap Z_1^+) \times (Z_1^- \cap Z_2^+) \arrow[rr, bend left, "\Psi"] & Z_0^- \cap \{z_1 \neq 0\} \cap Z_2^+ \arrow[l, "\sim"'] \arrow[r, hook, "\text{open}"] & Z_0^- \cap Z_2^+
\end{tikzcd}
$$

$$
\{(\underset{\neq 0}{z_0} : \underset{\neq 0}{z_1} : 0)\} \times \{(0 : \underset{\neq 0}{z_1} : z_2)\} \qquad \{(\underset{\neq 0}{z_0} : \underset{\neq 0}{z_1} : \underset{\neq 0}{z_2})\} \qquad \{(\underset{\neq 0}{z_0} : z_1 : \underset{\neq 0}{z_2})\}
$$

$$
((0:z_1:z_2),\ (z_0:z_1:0)) \longleftarrow\!\shortmid\ (z_0:z_1:z_2) \longmapsto (z_0:z_1:z_2)
$$

Part II

Details + Diagrams

Notation

$T \subset B \subset G$ — max. torus, Borel, reductive group

$U \subset B$ — unipotent

$w_0 \in W = N_G(T)/T$ — longest element, Weyl group

$\overline{B} = B^{w_0}$, $\overline{U} = U^{w_0}$ .... — opposite

$\Delta \subset \Phi^+ \subset \Phi \subset X(T)$ — simple, positive, roots, characters

$B \subset P \subset G \rightsquigarrow W_P \subset W$ — std. parabolic

$$\boxed{Z = G/P}$$

Bruhat = Białynicki-Birula stratification

$\eta \in Y(T)$ generic cocharacter with $\langle \eta, \alpha \rangle > 0 \ \forall \alpha \in \Phi^+$

$$\rightsquigarrow \quad \mathbb{G}_m \xrightarrow{\eta} T \hookrightarrow Z$$

$$Z^0 = \biguplus_{w \in W/W_P} wP/P \, , \qquad Z^+ = \biguplus_{w \in W/W_P} BwP/P \qquad Z^- = \biguplus_{w \in W/W_P} \overline{B}wP/P$$

Bruhat cells

$$Z_x^- \cap Z_y^+$$

open Richardson variety (first in Kazhdan-Lusztig)

Parametrizations

$$U_w = U \cap w\overline{U}w^{-1}, \qquad \overline{U}_w = \overline{U} \cap w U w^{-1}$$

Then

$$U_w \xrightarrow{\sim} U_w wB/P = Z_w^+, \qquad \overline{U}_w \xrightarrow{\sim} \overline{U}_w wB/P = Z_w^-$$

Obtain:

swap iso.

$$\begin{tikzcd} \overline{U}_w \times U_w \arrow[r, "\sim"] \arrow[rr, bend left, "\sigma_w"] & V_w \overset{\text{def}}{=} \overline{U}_w U_w wB/B = U_w \overline{U}_w wB/B & U_w \times \overline{U}_w \arrow[l, "\sim"'] \end{tikzcd}$$

open neighborhood of wP/P

⚠ σ_w is non-trivial already for GL_3/B ⚠

Multiplicative structure

$$\psi : Z_x^- \cap Z_y^+ \times Z_y^- \cap Z_z^+ \xrightarrow{\sim} Z_x^- \cap V_y \cap Z_z^+ \hookrightarrow Z_x^- \cap Z_z^+$$

$$(uyP/P \, , \, \bar{u}yP/P) \longmapsto \bar{u}'u'yP/P$$

where $\bar{u}' \in \bar{U}_y$ and $u' \in U_y$, s.t. $\bar{u}' u w P/P = u'\bar{u}P/P \iff$

$\sigma_y(\bar{u}', u) = (u', \bar{u})$

$\rightsquigarrow$ $\boxed{\text{Thm B}}$ $(E, \circ) \cong \left(\bigoplus_{x,y \in Z^0} H_c^\bullet(Z_x^- \cap Z_y^+), *_\psi \right)$

product using ψ

Proof idea: $V = \bigoplus V_w$

$\rightsquigarrow$ $\tilde{V} \hookrightarrow \tilde{Z}$ and $\tilde{V}$ can be trivialized!

Grassmannians

$$G = GL_n, \quad P = \begin{array}{cc} & \overbrace{\qquad}^{d} \\ d\{ & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \quad \rightsquigarrow \quad Z = G/P = Gr(d,n)$$

$$W_P = S_d \times S_{n-d} \subset S_n = W, \qquad W/W_P \cong \binom{[n]}{d} \ni I$$

d-element subset of $\{1, .., n\}$

Example: $I = \{1,2,4\} \subset \{1, .., 7\}$, $J = \{2,3,7\}$

$$Z_I^- \cong \begin{bmatrix} 1 & & x & & x & x & x \\ & 1 & x & & x & x & x \\ & & & 1 & x & x & x \end{bmatrix} \cong \mathbb{A}^{11} \qquad Z_J^+ \cong \begin{bmatrix} x & 1 & & & & & \\ x & & x & x & 1 & & \\ x & & x & x & & x & 1 \end{bmatrix} = \mathbb{A}^8$$

Grassmannians

Abbreviate $\quad Z(I,J) = Z_I^- \cap Z_J^+$

Base change map Let $\mathcal{G} = GL_d$

$$\varphi : \quad Z(I,J) \longrightarrow \mathcal{G}(I,J) \subset \mathcal{G}$$

$$W = \operatorname{rowspan} \underset{\mathbb{C}^{d\times n}}{\underset{\cap}{X}} \longmapsto (X_{[d]\times I})^{-1} X_{[d]\times J}$$

Thm (E.-Stroppel): $\varphi : Z(I,J) \to \mathcal{G}(I,J)$ is a vector bundle

Decompositions

Let $\mathbb{B}, \overline{\mathbb{B}} \subset \mathcal{G}$ standard Borel + its opposite, $\mathbb{W} = S_d$

$$Z(I,J)^w = \varphi^{-1}(\mathbb{B} w \overline{\mathbb{B}}) \qquad Z(I,J)_w = \varphi^{-1}(\mathbb{B} w \mathbb{B})$$

Then

$$Z(I,J) = \biguplus Z(I,J)^w \qquad Z(I,J) = \biguplus Z(I,J)_w$$

Gauss-type decomposition $\qquad$ Deodhar-type decomposition

$\boxed{\text{Thm C}}$ Deodhar-type decomposition is a stratification and

$Z(I,J)_w = \mathcal{G}_m^a \times \mathbb{A}^b$ for a, b appropriate

Proof idea: Reduce to Bruhat stratification of $\mathcal{G}$

Normal forms

Again let $I = \{1,2,4\} \subset \{1,\dots,7\}$, $J = \{2,3,7\}$

Denote $\mathbb{U} \subset \mathbb{B}$ unipotent matrices and $\mathbb{T} \subset \mathbb{B}$ max. torus. Let

$$M_I^- = \begin{bmatrix} 1 & & x & & x & x & x \\ & 1 & x & & x & x & x \\ & & & 1 & x & x & x \end{bmatrix} \subset k^{d \times n} \qquad M_J^- = \begin{bmatrix} x & 1 & & & & & \\ x & & x & x & 1 & & \\ x & & x & x & & x & 1 \end{bmatrix} \subset k^{d \times n}$$

Then

$$Z(I,J)_w \cong M_I^- \cap BwB\, M_J^+$$

!difficult equations!

Better

$$Z(I,J)_w \cong \mathbb{U} M_I^- \cap w \mathbb{T} \mathbb{U}_w M_J^+$$

simple equations

Diagrams

Let $s = (1,2) \in S_3 = W$

$$\mathcal{U} M_I^- = \begin{bmatrix} 1 & x & x & x & x & x & x \\ & 1 & x & x & x & x & x \\ & & & 1 & x & x & x \end{bmatrix} \qquad s\Pi\mathcal{U}_s M_J^- = \begin{bmatrix} x & & x & x & 1 & & \\ x & \otimes & * & * & x & & \\ x & & x & x & & x & \otimes \end{bmatrix}$$

x $\mathbb{A}^1$

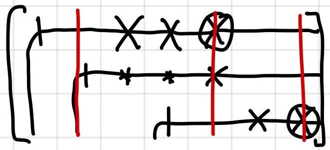
$\otimes$ $\mathbb{G}_m$

$*$ determined by rest of matrix

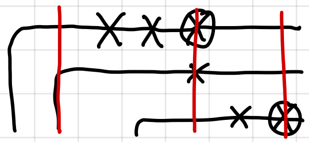
$$\rightsquigarrow \quad Z(I,J)_s \cong \mathcal{U} M_I^- \cap s\Pi\mathcal{U}_s M_J^- = \begin{bmatrix} 1 & & x & x & \otimes & & \\ & 1 & * & * & x & & \\ & & & 1 & & x & \otimes \end{bmatrix} \cong \mathbb{G}_m^2 \times \mathbb{A}^4$$

Diagrammatics

$$\begin{bmatrix} 1 & & x & x & \otimes & & \\ & 1 & * & * & x & & \\ & & & 1 & & x & \otimes \end{bmatrix} \rightsquigarrow$$

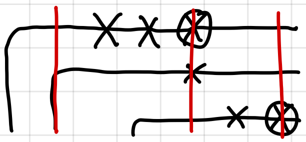


$\rightsquigarrow$



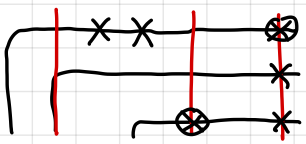
Deodhar type Fukaya diagram

$Z(I,J) =$



$\mathbb{G}_m^2 \times \mathbb{A}^4$

$\oplus$



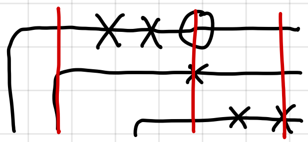
$\mathbb{G}_m^2 \times \mathbb{A}^5$

Diagrams

Works in general: Simple rule set to build all diagrams.

Encodes cohomology. $H_c^\bullet(\mathbb{A}^1) = \mathbb{Z}[-1]$ $\quad H_c^\bullet(\mathbb{G}_m) = \mathbb{Z}[-1] \oplus \mathbb{Z}[-2]$

deg $\times = 1$ $\quad$ deg $\times = 1$ $\quad$ deg $\bigcirc = 2$

$\rightsquigarrow$  $\in H_c^7(Z(I,J)_s)$

Combinatorics Diagrams have same combinatoris as parabolic Deodhar decomposition

Point counting

$R(q) = \# Z(I,J)$ over $\mathbb{F}_q$ $=$ Sum over point counts of strata $Z(I,J)_w$

Outlook

Thm. E Let $A = \bigoplus_{I,J} \bigoplus_{w} H_c^\bullet(Z(I,J)_w)$ ← related to Nil-Coxeter algebra

Then A has a differential d, s.t.,

$$E = H^\bullet(A, d)$$

Proof Sketch A is a page in a spectral sequence that degenerates for weight reasons □

Conj.: There is a multiplication on A, s.t. (A, d) is a dg-model for E.

Cor. of Conj $D_{Z^+}(Z) \cong D_{perf}(A, d)$

⋮

THANKS!